

The construction of the program control with probability one for stochastic dynamic systems with jumps

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Abstract

Investigate the stochastic dynamic non-linear system with the Wiener and the Poisson perturbations:

$$d\mathbf{x}(t) = \left(P(t; \mathbf{x}(t)) + Q(t; \mathbf{x}(t)) \cdot \mathbf{s}(t; \mathbf{x}(t)) \right) dt + \varepsilon \left(t; \mathbf{x}(t), d\mathbf{w}(t), \nu(dt; d\gamma) \right), \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$; $\varepsilon(\cdot)$ – is a strong random perturbation; $\mathbf{w}(t)$ – is the m -dimensional Wiener process; $\nu(t; \Delta\gamma)$ – is the homogeneous on t non-centered Poisson measure. For such systems we construct the program control $\mathbf{s}(t; \mathbf{x}(t))$ with probability one, which allows the system (1) to move on the given trajectory: $u_i(t; \mathbf{x}(t)) = 0$, $i = \overline{1, k}$. Control program $\mathbf{s}(t; \mathbf{x}(t))$ is solution of the algebraic system of linear equations. Considered algorithm is based on the first integral theory for stochastic differential equations system.¹

1 Preliminary definitions and results

Suppose that a dynamic system is affected by random disturbance, which takes the form of Wiener and Poisson perturbations. In such situation we require that important system properties are kept invariant. We investigate

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the program control construction problem with probability one for the similar systems. We use the first integral concept for the Itô's stochastic differential equation system (SDES) [1, 2, 3] and the automorphic function construction [4, 5].

Let us consider the random process $\mathbf{x}(\mathbf{t}) = \mathbf{x}(\mathbf{t}, \omega)$, $\mathbf{x} : [\mathbf{0}; +\infty) \rightarrow \mathbb{R}_+^{\mathbf{n}}$ which is the solution of the Cauchy problem for system of the Itô's generation stochastic differential equations (GSPE)

$$\begin{aligned} dx_i(t) &= a_i(t; \mathbf{x}(t))dt + \sum_{k=1}^m b_{ik}(t; \mathbf{x}(t))dw_k(t) + \int_{R(\gamma)} g_i(t; \mathbf{x}(t); \gamma)\nu(dt; d\gamma) \\ \mathbf{x}(t) &= \mathbf{x}(t, \mathbf{x}_0) \Big|_{t=0} = \mathbf{x}_0, \quad i = \overline{1, n}, \quad t \geq 0, \end{aligned} \quad (2)$$

where $\mathbf{w}(t)$ is the m -dimensional Wiener process, $\nu(t; \Delta\gamma)$ which is homogeneous on t non centered Poisson measure. This equation we present in the vector form

$$d\mathbf{x}(t) = A(t; \mathbf{x}(t))dt + B(t; \mathbf{x}(t))d\mathbf{w}(t) + \int_{R(\gamma)} \nu(dt; d\gamma) \cdot G(t; \mathbf{x}(t); \gamma)$$

We suppose that the coefficients $a(t; \mathbf{x})$, $b(t; \mathbf{x})$, and $g(t; \mathbf{x}; \gamma)$ are bounded and continuous together with their own derivations $\frac{\partial a_i(t; \mathbf{x})}{\partial x_j}$, $\frac{\partial b_{i,k}(t; \mathbf{x})}{\partial x_j}$, $\frac{\partial g_i(t; \mathbf{x}; \gamma)}{\partial x_j}$, $i, j = \overline{1, n}$ respect to $(t; \mathbf{x}; \gamma)$, and they are satisfied the conditions of the theorem of existence and uniqueness of the Eq. (2).

In the article [1] the concept of the first integral for Itô's GSDE system (without Poisson part) has been introduced. In the article [3] the concept of the stochastic first integral for Itô's GSDE system with the centered measure has been given. Let's introduce the similar concept for the non-centered Poisson measure.

Definition 1 *Let random function $u(t; \mathbf{x}; \omega)$ and solution $\mathbf{x}(t)$ of the Itô's GSDE system (2) be defined on the same probability space. The function $u(t; \mathbf{x}; \omega)$ is called the stochastic first integral for Itô's GSDE system (2) with non centered Poisson measure, if condition*

$$u\left(t; \mathbf{x}(t; \mathbf{x}(0)); \omega\right) = u\left(0; \mathbf{x}(0); \omega\right)$$

is held with probability equaled to 1 for the each solution $\mathbf{x}(t; \mathbf{x}(0); \omega)$ of the system (2).

The random function $u(t; \mathbf{x}(t)) \in \mathcal{C}_{t,x}^{1,2}$ defined over the same probability space as the random process $\mathbf{x}(t)$ which is the solution for the system (2) is the first integral of the system (2) iff the function $u(t; \mathbf{x}(t))$ satisfies the terms \mathcal{L} [6]:

1. $b_{ik}(t; \mathbf{x}) \frac{\partial u(t; \mathbf{x})}{\partial x_i} = 0$, for all $k = \overline{1, m}$ (compensation of the Wiener's perturbations);
2. $\frac{\partial u(t; \mathbf{x})}{\partial t} + \frac{\partial u(t; \mathbf{x})}{\partial x_i} \left[a_i(t; \mathbf{x}) - \frac{1}{2} b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} \right] = 0$ (independence of the time);
3. $u(t; \mathbf{x}) - u(t; \mathbf{x} + g(t; \mathbf{x}; \gamma)) = 0$ for any $\gamma \in R(\gamma)$ in the whole field of the process definition (compensation of the Poisson's jumps).

Remark 1 *If we analyze the concrete realization then the non random function $u(t; \mathbf{x})$ is the determinate first integral of the stochastic system.*

The concept of the stochastic first integral for the centered Poisson measure was introduced in [3]. The obtained conditions for its realization take the necessity of determining the intensive Poisson distribution density. In this research that condition is missing. Thus, it makes no difference what is the probability distribution of intensities of Poisson jumps. This case is very important for constructing program controls [5]. In case a Wiener disturbances only, we construct the program control with probability one in article [7] on basis the research [2].

Theorem 1 [5] *Let the determinate function $u(t, \mathbf{x})$ be continuous together with their own derivatives with respect to all variables $u(t, \mathbf{x})$ and the random function $u(t, \mathbf{x}; \omega)$ defined over the same probability space as the random process $\mathbf{x}(t)$ which is the solution for system*

$$\begin{aligned}
 d\mathbf{x}(t) &= A(t; \mathbf{x}(t))dt + B(t; \mathbf{x}(t))d\mathbf{w}(t) + \int_{R(\gamma)} \nu(dt; d\gamma) \cdot G(t; \mathbf{x}(t); \gamma) \\
 \mathbf{x}(t) &= \mathbf{x}(t, \mathbf{x}_0) \Big|_{t=0} = \mathbf{x}_0, \quad t \geq 0,
 \end{aligned} \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$; $\mathbf{w}(t)$ is an m -dimensional Wiener process, $\nu(t; \Delta\gamma)$ – is homogeneous on t non centered Poisson measure. Assume the vectors \vec{e}_0 , $\vec{e}_1, \dots, \vec{e}_n$ consist an orthogonal basis in $\mathbb{R}_+ \times \mathbb{R}^n$. If function $u(t, \mathbf{x}; \omega)$ is the first integral of the system (3), then the coefficients of an Eq. (3) and the function $u(t, \mathbf{x})$ are related by conditions:

1. coefficients $B_k(t; \mathbf{x}) = \sum_{i=1}^n b_{ik}(t; \mathbf{x}) \vec{e}_i$ ($k = \overline{1, m}$), are columns of the matrix $B(t; \mathbf{x})$, that belongs to a set of functions

$$B_k(t; \mathbf{x}) \in \left\{ q_{oo}(t; \mathbf{x}) \cdot \det \begin{pmatrix} \vec{e}_1 & \dots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{x})}{\partial x_1} & \dots & \frac{\partial u(t; \mathbf{x})}{\partial x_n} \\ f_{31} & \dots & f_{3n} \\ \dots & \dots & \dots \\ f_{n1} & \dots & f_{nn} \end{pmatrix} \right\}, \quad (4)$$

where $q_{oo}(t; \mathbf{x})$ is an arbitrary non-vanishing function;

2. coefficient $A(t; \mathbf{x})$ belongs to a set of the functions, defined by

$$A(t; \mathbf{x}) \in \left\{ R(t; \mathbf{x}) + \frac{1}{2} \sum_{k=1}^n \left[\frac{\partial B_k(\cdot)}{\partial \mathbf{x}} \right] \cdot B_k(\cdot) \right\}, \quad (5)$$

where a column matrix $R(t; \mathbf{x})$ with components $r_i(t; \mathbf{x})$, $i = \overline{1, n}$, are defined by:

$$C^{-1}(t; \mathbf{x}) \cdot \det H(t; \mathbf{x}) = \vec{e}_0 + \sum_{i=1}^n r_i(t; \mathbf{x}) \vec{e}_i;$$

$C(t; \mathbf{x})$ is an algebraic adjunct of the element \vec{e}_0 of a matrix $H(t; \mathbf{x})$ and $\det C(t; \mathbf{x}) \neq 0$; a matrix $H(t; \mathbf{x})$ is defined as

$$H(t; \mathbf{x}) = \begin{bmatrix} \vec{e}_0 & \vec{e}_1 & \dots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{x})}{\partial t} & \frac{\partial u(t; \mathbf{x})}{\partial x_1} & \dots & \frac{\partial u(t; \mathbf{x})}{\partial x_n} \\ h_{30} & h_{31} & \dots & h_{3n} \\ \dots & \dots & \dots & \dots \\ h_{n+1,0} & h_{n+1,1} & \dots & h_{n+1,n} \end{bmatrix}, \quad (6)$$

and $\left[\frac{\partial B_k(t; \mathbf{x})}{\partial \mathbf{x}} \right]$ is a matrix of Jacobi for function $B_k(t; \mathbf{x})$;

3. coefficient $G(t; \mathbf{x}; \gamma) = \sum_{i=1}^n g_i(t; \mathbf{x}; \gamma) \vec{e}_i$ belonging to Poisson measure, is defined by the next representation $G(t; \mathbf{x}; \gamma) = \mathbf{y}(t; \mathbf{x}; \gamma) - \mathbf{x}$, where $\mathbf{y}(t; \mathbf{x}; \gamma)$ is the solution of the differential equations system

$$\begin{aligned} & \frac{\partial \mathbf{y}(\cdot; \gamma)}{\partial \gamma} = \\ & = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_1} & \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_2} & \cdots & \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_n} \\ \varphi_{31}(t; \mathbf{y}(\cdot; \gamma)) & \varphi_{32}(t; \mathbf{y}(\cdot; \gamma)) & \cdots & \varphi_{3n}(t; \mathbf{y}(\cdot; \gamma)) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1}(t; \mathbf{y}(\cdot; \gamma)) & \varphi_{n2}(t; \mathbf{y}(\cdot; \gamma)) & \cdots & \varphi_{nn}(t; \mathbf{y}(\cdot; \gamma)) \end{bmatrix}. \end{aligned} \quad (7)$$

This solution satisfies the initial conditions: $\mathbf{y}(t; \mathbf{x}; \gamma) \Big|_{\gamma=0} = \mathbf{x}$.

The arbitrary functions $f_{ij} = f_{ij}(t, \mathbf{x})$, $h_{ij} = h_{ij}(t, \mathbf{x})$, $\varphi_{ij} = \varphi_{ij}(t; \mathbf{y}(\cdot; \gamma))$ are defined by the equalities $f_{ij}(t, \mathbf{x}) = \frac{\partial f_i(t, \mathbf{x})}{\partial x_j}$, $h_{ij}(t, \mathbf{x}) = \frac{\partial h_i(t, \mathbf{x})}{\partial x_j}$, $\varphi_{ij}(t; \mathbf{y}(\cdot; \gamma)) = \frac{\partial \varphi_i(t; \mathbf{y}(\cdot; \gamma))}{\partial y_j}$, and sets of the functions $\{f_i\}$, $\{h_i\}$, $\{\varphi_i\}$ and the function $u(t; \mathbf{x})$ together consist of the class of independent functions.

Proof. We consider three steps for the proof.

1. Let us use the first statement from the conditions \mathcal{L} :

$$\sum_{i=1}^n b_{ik}(t; \mathbf{x}) \frac{\partial u(t; \mathbf{x})}{\partial x_i} = 0, \quad \text{for all } k = \overline{1, m}. \quad (8)$$

If $B_k(t; \mathbf{x}) = \sum_{i=1}^n b_{ik}(t; \mathbf{x}) \vec{e}_i$ and $\nabla_{\mathbf{x}} u(t; \mathbf{x}) = \sum_{i=1}^n \frac{\partial u(t; \mathbf{x})}{\partial x_i} \vec{e}_i$ hold, then Eq. (8) is an orthogonal property of vectors $B_k(t; \mathbf{x})$ and $\nabla_{\mathbf{x}} u(t; \mathbf{x})$.

Taking into account a vector product definition in \mathbb{R}^n and their properties

we set a representation for columns $B_k(t; \mathbf{x})$ of matrix $B(\cdot) = (B_1(\cdot), \dots, B_m(\cdot))$:

$$B_k(t; \mathbf{x}) \in \left\{ q_{oo}(t; \mathbf{x}) \cdot \det \begin{pmatrix} \vec{e}_1 & \dots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{x})}{\partial x_1} & \dots & \frac{\partial u(t; \mathbf{x})}{\partial x_n} \\ f_{31} & \dots & f_{3n} \\ \dots & \dots & \dots \\ f_{n1} & \dots & f_{nn} \end{pmatrix} \right\}; \quad (4)$$

where $f_i = f_i(t, \mathbf{x})$, $i = \overline{3, n}$, and functions $f_{ij}(t, \mathbf{x}) = \frac{\partial f_i(t, \mathbf{x})}{\partial x_j}$ and $u(t, \mathbf{x})$ together generate the collection of a independence ones.

2. Later we use the second statement of the conditions \mathcal{L} :

$$\frac{\partial u(t; \mathbf{x})}{\partial t} + \sum_{i=1}^n \frac{\partial u(t; \mathbf{x})}{\partial x_i} \left[a_i(t; \mathbf{x}) - \frac{1}{2} \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} \right] = 0. \quad (9)$$

Let us suppose

$$Q(t; \mathbf{x}) = 1 + \sum_{i=1}^n a_i(t; \mathbf{x}) - \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j}.$$

According to the scheme in article [2] we consider two vectors:

$$\square u(t; \mathbf{x}) = \frac{\partial u(t; \mathbf{x})}{\partial t} \vec{e}_0 + \sum_{i=1}^n \frac{\partial u(t; \mathbf{x})}{\partial x_i} \vec{e}_i$$

and

$$\vec{Q}(t; \mathbf{x}) = \vec{e}_0 + \sum_{i=1}^n a_i(t; \mathbf{x}) \vec{e}_i - \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} \vec{e}_i.$$

Then the Eq. (9) means that the vectors $\square u(t; \mathbf{x})$ and $\vec{Q}(t; \mathbf{x})$ are orthogonal. Later we use the vector product and it's properties again, and we obtain:

$$\vec{Q}(t; \mathbf{x}) \in \left\{ \det \begin{bmatrix} \vec{e}_0 & \vec{e}_1 & \dots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{x})}{\partial t} & \frac{\partial u(t; \mathbf{x})}{\partial x_1} & \dots & \frac{\partial u(t; \mathbf{x})}{\partial x_n} \\ h_{30} & h_{31} & \dots & h_{3n} \\ \dots & \dots & \dots & \dots \\ h_{n+1,0} & h_{n+1,1} & \dots & h_{n+1,n} \end{bmatrix} \right\} = \{\det H\}, \quad (10)$$

where $h_i = f_i(t, \mathbf{x})$, $i = \overline{3, n+1}$, and functions $h_{ij}(t, \mathbf{x}) = \frac{\partial h_i(t, \mathbf{x})}{\partial x_j}$ and $u(t, \mathbf{x})$ together generate the collection of a independence ones. Let us consider the vector

$$\vec{A}(t; \mathbf{x}) = \vec{e}_o + \sum_{i=1}^n a_i(t; \mathbf{x}) \vec{e}_i = \vec{Q}(t; \mathbf{x}) + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} \vec{e}_i.$$

As a coefficient of \vec{e}_o is equal to 1, then we get:

$$\vec{A}(t; \mathbf{x}) \in \left\{ C^{-1}(t; \mathbf{x}) \cdot \det H(t; \mathbf{x}) + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} \vec{e}_i \right\},$$

where $C(t; \mathbf{x})$ is an algebraical adjunct of element \vec{e}_o of matrix $H(t; \mathbf{x})$, $\det C(t; \mathbf{x}) \neq 0$. As far as the vector $C^{-1}(t; \mathbf{x}) \cdot \det H(t; \mathbf{x})$ we present in a form:

$$C^{-1}(t; \mathbf{x}) \cdot \det H(t; \mathbf{x}) = \vec{e}_o + \sum_{i=1}^n r_i(t; \mathbf{x}) \vec{e}_i,$$

let us introduce the following vector $R(t; \mathbf{x})$ that has components $r_i(t; \mathbf{x})$, $i = \overline{1, n}$.

We might the next representation:

$$\sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^n b_{jk}(t; \mathbf{x}) \frac{\partial b_{ik}(t; \mathbf{x})}{\partial x_j} = \sum_{k=1}^m \left[\frac{\partial B_k(t; \mathbf{x})}{\partial \mathbf{x}} \right] \cdot B_k(t; \mathbf{x}), \quad (11)$$

where $\left[\frac{\partial B_k(t; \mathbf{x})}{\partial \mathbf{x}} \right]$ is a matrix of Jacobi for function $B_k(t; \mathbf{x})$. It follows that a coefficient $A(t; \mathbf{x})$ is a sum of matrices:

$$A(t; \mathbf{x}) \in \left\{ R(t; \mathbf{x}) + \frac{1}{2} \sum_{k=1}^m \sum_{k=1}^n \left[\frac{\partial B_k(t; \mathbf{x})}{\partial \mathbf{x}} \right] \cdot B_k(t; \mathbf{x}) \right\},$$

3. According to the third statement in \mathcal{L} for all $\gamma \in R(\gamma)$ we have:

$$u(t; \mathbf{x}; \omega) - u\left(t; \mathbf{x} + G(t; \mathbf{x}; \gamma); \omega\right) = 0. \quad (12)$$

It means that the function $u(t; \mathbf{x}; \omega)$ is automorphic function under translation by \mathbf{x} with function $G(t; \mathbf{x}; \gamma)$. Let us set a condition for it.

According to [3] we have $\mathbf{y}(t; \mathbf{x}; \gamma) = \mathbf{x} + G(t; \mathbf{x}; \gamma)$. For notational simplicity we take off a parameter ω . Then we obtain

$$u(t; \mathbf{x}) = u\left(t; \mathbf{y}(t; \mathbf{x}; \gamma)\right) \quad (13)$$

for all $\gamma \in R(\gamma)$. Hence, $\frac{\partial u(t; \mathbf{x})}{\partial \gamma} = 0$ and it holds:

$$\frac{\partial u\left(t; \mathbf{y}(t; \mathbf{x}; \gamma)\right)}{\partial \gamma} \equiv \sum_{i=1}^n \frac{\partial u\left(t; \mathbf{y}(t; \mathbf{x}; \gamma)\right)}{\partial y_i} \frac{\partial y_i(t; \mathbf{x}; \gamma)}{\partial \gamma} = 0. \quad (14)$$

Eq. p denotes that the vectors $\nabla_{\mathbf{y}} u\left(t; \mathbf{y}(\cdot; \gamma)\right) = \sum_{i=1}^n \frac{\partial u\left(t; \mathbf{y}(\cdot; \gamma)\right)}{\partial y_i} \vec{e}_i$ and $\frac{\partial \mathbf{y}(\cdot; \gamma)}{\partial \gamma} = \sum_{i=1}^n \frac{\partial y_i(\cdot; \gamma)}{\partial \gamma} \vec{e}_i$ are orthogonal and they connected by the relation

$$\frac{\partial \mathbf{y}(\cdot; \gamma)}{\partial \gamma} \in \left\{ \det \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_n \\ \frac{\partial u\left(t; \mathbf{y}(\cdot; \gamma)\right)}{\partial y_1} & \cdots & \frac{\partial u\left(t; \mathbf{y}(\cdot; \gamma)\right)}{\partial y_n} \\ \varphi_{31} & \cdots & \varphi_{3n} \\ \cdots & \cdots & \cdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{bmatrix} \right\}. \quad (15)$$

There are $\varphi_i(t; \mathbf{y})$, $i = \overline{3, n}$ such functions that $\varphi_{ij} = \frac{\partial \varphi_i(t; \mathbf{y})}{\partial y_j}$ and $u\left(t; \mathbf{y}(\cdot; \gamma)\right)$ together generate the collection of a independence functions.

By virtue of the fact that $\mathbf{y}(t; \mathbf{x}; \gamma) = \mathbf{x} + G(t; \mathbf{x}; \gamma)$ then the Eq. (15) is the differential system, which has a function $\mathbf{y}(\cdot; \gamma)$ as unknown. Let us expand the determinate (15) into the first row. Then we have $\frac{\partial \mathbf{y}(\cdot; \gamma)}{\partial \gamma} =$

$\alpha \sum_{i=1}^n S_i(\mathbf{y}(\cdot; \gamma)) \vec{e}_i$, where α is an arbitrary function, which independent from \mathbf{y} .

Such away we obtain the next differential system:

$$\begin{cases} \frac{\partial y_1(\cdot; \gamma)}{\partial \gamma} = \alpha S_1(\mathbf{y}(\cdot; \gamma)), \\ \cdots \\ \frac{\partial y_n(\cdot; \gamma)}{\partial \gamma} = \alpha S_n(\mathbf{y}(\cdot; \gamma)). \end{cases} \quad (16)$$

Suppose that $\mathbf{y}(t; \mathbf{x}; \gamma; \theta)$ is a solution of the Eq. (16), where θ is a constant vector. By virtue of (13) holds for all t , \mathbf{x} and for all γ , then is

$$u(t; \mathbf{x}) = u\left(t; \mathbf{y}(t; \mathbf{x}; \gamma_1; \theta)\right) = u\left(t; \mathbf{x} + G(t; \mathbf{x}; \gamma_1; \theta)\right) = u\left(t; \mathbf{x} + G(t; \mathbf{x}; \gamma_2; \theta)\right). \quad (17)$$

As a special case for same $\gamma = \gamma_o$ the Eq. (17) is determined by expression $G(t; \mathbf{x}; \gamma_o; \theta) = 0$.

Unconstrained in common we suppose that

$$G(t; \mathbf{x}; \gamma_o; \theta) \equiv G(t; \mathbf{x}; 0) = 0.$$

Such away, the system (13) which has an initial conditions $\mathbf{y}(t; \mathbf{x}; \gamma)\big|_{\gamma=0} = \mathbf{x}$ and it has an uniqueness solving. This system we transform to the form:

$$\frac{\partial \mathbf{y}(\cdot; \gamma)}{\partial \gamma} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_1} & \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_2} & \cdots & \frac{\partial u(t; \mathbf{y}(\cdot; \gamma))}{\partial y_n} \\ \varphi_{31}(t; \mathbf{y}(\cdot; \gamma)) & \varphi_{32}(t; \mathbf{y}(\cdot; \gamma)) & \cdots & \varphi_{3n}(t; \mathbf{y}(\cdot; \gamma)) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1}(t; \mathbf{y}(\cdot; \gamma)) & \varphi_{n2}(t; \mathbf{y}(\cdot; \gamma)) & \cdots & \varphi_{nn}(t; \mathbf{y}(\cdot; \gamma)) \end{bmatrix}. \quad (7)$$

Consequently, the function $G(t; \mathbf{x}; \gamma)$ is automorphic transform for the function $u(t; \mathbf{x})$ and it has the next representation: $G(t; \mathbf{x}; \gamma) = \mathbf{y}(t; \mathbf{x}; \gamma) - \mathbf{x}$, where $\mathbf{y}(t; \mathbf{x}; \gamma)$ is a solution of the differential system (7) with an initial condition $\mathbf{y}(t; \mathbf{x}; \gamma)\big|_{\gamma=0} = \mathbf{x}$.

2 Construction of the program control

On the analogy of the article [7] let us introduce the following definition.

Definition 2 *By the program motion of the stochastic system*

$$\begin{aligned} d\mathbf{x}(t) = & \left[P(t; \mathbf{x}(t)) + Q(t; \mathbf{x}(t)) \cdot \mathbf{s}(t; \mathbf{x}(t)) \right] dt + \\ & + B(t; \mathbf{x}(t)) d\mathbf{w}(t) + \int_{R(\gamma)} \nu(dt; d\gamma) G(t; \mathbf{x}(t); \gamma), \end{aligned} \quad (18)$$

where $\mathbf{w}(t)$ is the m -dimensional Wiener process; $\nu(t; \Delta\gamma)$ – is homogeneous on t non centered Poisson measure, is meant the solution $\mathbf{x}(t; \mathbf{x}_o, \mathbf{s}; \omega)$ which enables one under some (program) control $\mathbf{s}(t; \mathbf{x})$ and for all t to remain with the probability 1 on the given integral manifold $u(t; \mathbf{x}(t; \mathbf{x}_o)) = u(0; \mathbf{x}_o)$, which is the first integral of the Eq. (18) under the given initial conditions

$$\mathbf{x}(t; \mathbf{x}_o) \Big|_{t=0} = \mathbf{x}_o.$$

By this means we can construct the program control with the probability equaled to 1 for dynamic systems which are subjected to the Wiener perturbations and the Poisson jumps.

Theorem 2 *The control $\mathbf{s}(t; \mathbf{x})$ allowing system (18) which is subjected to the Wiener perturbations and the Poisson jumps always remain on the dynamically structured integral manifold $u(t; \mathbf{x}(t; \mathbf{x}_o); \omega) = u(0; \mathbf{x}_o)$ is determined as the solution of the system of linear equation, which consists of the Eq. (18) and the Eq. (3). The coefficients of the second equation and respectively coefficient of the first equation of this system are determined by the Theorem 1. In addition we completely define the response to random action.*

Example 1 *Needed is to determine the control $\mathbf{s}(t; \mathbf{x})$ and the response to random action enabling to dynamical system*

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \left(x_1(t) + x_2(t) + e^{-t} + s_1(t; \mathbf{x}(t)) \right) dt + \\ &+ b_1(t; \mathbf{x}(t)) dw(t) + \int_{R(\gamma)} g_1(t; \mathbf{x}(t); \gamma) \nu(dt; d\gamma), \\ \frac{dx_2(t)}{dt} &= \left(x_1(t)x_2(t) + e^{-2t} + s_2(t; \mathbf{x}(t)) \right) dt + \\ &+ b_2(t; \mathbf{x}(t)) dw(t) + \int_{R(\gamma)} g_2(t; \mathbf{x}(t); \gamma) \nu(dt; d\gamma), \end{aligned} \tag{19}$$

to remain on the given integral manifold $u(t; \mathbf{x}(t)) = x_2(t)e^{-2x_1(t)}$ with probability one.

Solution. According to the Theorem 2 we construct the GSDE system, which has a function $u(t; \mathbf{x}) = x_2e^{-2x_1}$ as the first integral.

We first find an automorphic transformation of function $u(t; \mathbf{x}) = x_2e^{-2x_1}$. The corresponding partial derivatives of function $u(t; \mathbf{y}) = y_2e^{-2y_1}$ are given

by

$$\frac{\partial u(\mathbf{y}; t)}{\partial y_1} = -2y_2 e^{-2y_1}, \quad \frac{\partial u(\mathbf{y}; t)}{\partial y_2} = e^{-2y_1}. \quad (20)$$

Then we have

$$\frac{\partial \mathbf{y}(t; \mathbf{x}; \gamma)}{\partial \gamma} \equiv \begin{pmatrix} \frac{\partial y_1(t; \mathbf{x}; \gamma)}{\partial \gamma} \\ \frac{\partial y_2(t; \mathbf{x}; \gamma)}{\partial \gamma} \end{pmatrix} = \begin{pmatrix} e^{-2y_1} \\ 2y_2 e^{-2y_1} \end{pmatrix}.$$

Solution of this system is:

$$\begin{aligned} y_1(t; \mathbf{x}; \gamma) &= \frac{1}{2} \ln(2\gamma + 2C_1(\mathbf{x})) \\ y_2(t; \mathbf{x}; \gamma) &= C_2(\mathbf{x}) (\gamma + C_1(\mathbf{x})). \end{aligned}$$

With regard to initial conditions (according to Theorem 1, point 3), i. e. $\mathbf{y}(t; \mathbf{x}; \gamma)|_{\gamma=0} = \mathbf{x}$, we obtain:

$$C_1(\mathbf{x}) = \frac{1}{2} e^{2x_1}, \quad C_2(\mathbf{x}) = 2x_2 e^{-2x_1}.$$

And thus we have:

$$\begin{aligned} y_1(t; \mathbf{x}; \gamma) &= \frac{1}{2} \ln(2\gamma + e^{2x_1}), \\ y_2(t; \mathbf{x}; \gamma) &= 2x_2 \gamma e^{-2x_1} + x_2. \end{aligned}$$

Consequently, automorphic transformation $g(\cdot) = (g_1(\cdot), g_2(\cdot))^*$ of the function $u(t; \mathbf{x}) = x_2 e^{-2x_1}$ is:

$$\begin{aligned} g_1(t; \mathbf{x}; \gamma) &= \frac{1}{2} \ln(2\gamma + e^{2x_1}) - x_1, \\ g_2(t; \mathbf{x}; \gamma) &= 2x_2 \gamma e^{-2x_1}. \end{aligned} \quad (21)$$

According to the statement 1 of the Theorem 1 we construct a matrix B (in this case it is a column as far as $\mathbf{w}(t)$ is one-dimensional Wiener process):

$$B(\cdot) \equiv B(t; \mathbf{x}) = q_{oo} (e^{-2x_1}, 2x_2 e^{-2x_1})^*,$$

where $q_{oo} = q_{oo}(t; \mathbf{x})$. Let us define the second expression in Eq. (5):

$$\left[\frac{\partial B(t; \mathbf{x})}{\partial \mathbf{x}} \right] = q_{oo} \begin{pmatrix} -2e^{-2x_1} & 0 \\ 4x_2 e^{-2x_1} & 2e^{-2x_1} \end{pmatrix},$$

$$\left[\frac{\partial B(t; \mathbf{x})}{\partial \mathbf{x}} \right] B(t; \mathbf{x}) = q_{oo}^2 \begin{pmatrix} -4e^{-2x_1} \\ 0 \end{pmatrix} = \begin{pmatrix} -4q_{oo}^2 e^{-4x_1} \\ 0 \end{pmatrix}.$$

By Eq. (6) let's construct the matrix $H(t; \mathbf{x})$ and their determinant

$$\begin{aligned} \det H(t; \mathbf{x}) &= \det \begin{pmatrix} \vec{e}_0 & \vec{e}_1 & \vec{e}_2 \\ 0 & -2x_2 e^{-2x_1} & e^{-2x_1} \\ f_1 & f_2 & f_3 \end{pmatrix} = \\ &= \vec{e}_0 \left(-2f_3 x_2 e^{-2x_1} - f_2 e^{-2x_1} \right) + \vec{e}_1 \left(f_1 e^{-2x_1} \right) + \\ &\quad + \vec{e}_2 \left(2f_1 x_2 e^{-2x_1} \right), \end{aligned}$$

where $f_i = f_i(t; \mathbf{x})$, $i = 1, 2, 3$.

In result the components of vector $A = A(t; \mathbf{x})$ are given by

$$\begin{aligned} a_1 &= -\frac{f_1}{f_2 + 2f_3 x_2} + 2q_{oo}^2 e^{-4x_1}, \\ a_2 &= -\frac{2f_1 x_2}{f_2 + 2f_3 x_2}. \end{aligned}$$

Hence, the desired GSDE system is:

$$\begin{aligned} dx_1(t) &= \left[-\frac{f_1}{f_2 + 2f_3 x_2(t)} + 2q_{oo}^2 e^{-4x_1(t)} \right] dt + \\ &+ q_{oo} e^{-2x_1(t)} dw(t) + \int_{R(\gamma)} \left(\frac{1}{2} \ln(2\gamma + e^{2x_1(t)}) - x_1(t) \right) \nu(dt; d\gamma) \\ dx_2(t) &= \left[-\frac{2f_1 x_2(t)}{f_2 + 2f_3 x_2(t)} \right] dt + \\ &+ q_{00} 2x_2(t) e^{-2x_1(t)} dw(t) + \int_{R(\gamma)} (2x_2(t) \gamma e^{-2x_1(t)}) \nu(dt; d\gamma) \end{aligned}$$

Then we must solve the system of linear equation

$$\begin{aligned} x_1(t) + x_2(t) + e^{-t} + s_1(t; \mathbf{x}(t)) &= -\frac{f_1}{f_2 + 2f_3 x_2(t)} + 2q_{oo}^2 e^{-4x_1(t)}, \\ x_1(t)x_2(t) + e^{-2t} + s_2(t; \mathbf{x}(t)) &= -\frac{2f_1 x_2(t)}{f_2 + 2f_3 x_2(t)}. \end{aligned}$$

Hence, program control is of the form of

$$\begin{aligned} s_1(t; \mathbf{x}(t)) &= -\frac{f_1}{f_2 + 2f_3 x_2(t)} + 2q_{oo}^2 e^{-4x_1} - x_1(t) - x_2(t) - e^{-t}, \\ s_2(t; \mathbf{x}(t)) &= -\frac{2f_1 x_2(t)}{f_2 + 2f_3 x_2(t)} - x_1(t)x_2(t) - e^{-2t}, \end{aligned}$$

where $f_i = f_i(t; \mathbf{x}(t))$, $i = 1, 2, 3$, $q_{oo} = q_{oo}(t; \mathbf{x}(t))$ and $f_2 + 2f_3x_2(t) \neq 0$. The response to the Wiener action is defined as

$$b_1(t; \mathbf{x}(t)) = q_{oo}e^{-2x_1(t)}, \quad b_2(t; \mathbf{x}(t)) = q_{oo}2x_2(t)e^{-2x_1(t)}.$$

Components of the response to the Poisson jumps are defined as:

$$g_1(t; \mathbf{x}(t); \gamma) = \frac{1}{2} \ln (2\gamma + e^{2x_1(t)}) - x_1(t),$$

$$g_2(t; \mathbf{x}(t); \gamma) = 2x_2(t)\gamma e^{-2x_1(t)}.$$

Therefore, the set of vectors of the program controls and their corresponding responses to a random action enabling the dynamical system under consideration to remain on the given time-variable manifold are determined.

The choice of functions $f_i(t; \mathbf{x}(t))$, $i = 1, 2, 3$ and $q_{oo}(t; \mathbf{x}(t))$ allows to construct the program control on account of some conditions. For example we take into account a simulation utility.

Remark 2 *It is worthy to note that the manifold can be defined by a set of functions [6].*

Conclusion

Application of the first integral stochastic theory allows to construct the program control for dynamic system with probability equaled to 1 of there are strong random perturbations, caused by Wiener and Poisson processes.

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